Multiple invasion percolation

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Abstract

We generalize the standard site invasion percolation model to permit simultaneous invasion of several sites. We propose two kinds of generalizations: one in which the invasion flux is controlled by the perimeter size and another where the growth process is commanded by the scaling properties. The acceptance profile as well as the fractal dimensions $D_F$ are carefully studied. For the model based on scaling relations, $D_F$ can be treated as a mere real parameter in the range $(0, \infty)$. In the intervals $(0, \frac{91}{48})$ and $(2, \infty)$ the system is frustrated. For $D_F > 2$ the model exhibits also an interesting burst phenomenon which is explained in the text. In the region $[\frac{91}{48}, 2]$, the clusters obey exactly and in any scale the relation $M \sim R_g^{D_F}$ between the mass $M$ and the gyration radius $R_g$. These stabilized random fractals may be very useful in the study of dilute systems.

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1. Introduction

Invasion percolation is a dynamic percolation process introduced by Wilkinson and Willemsen [1] motivated by the study of the flow of two immiscible fluids in porous media. Usually, a nonviscous liquid is injected into a porous medium already filled with a viscous fluid. Depending on the injection rate, this system can be found in two different regimes: one where the dominant forces are of capillary nature and another where the viscous forces are predominant. The theoretical description of such a system is mainly based on two models: the invasion percolation [1] and the diffusion-limited aggregation (DLA) [2]. Invasion percolation is more appropriate when the fluid flow is slow, i.e., when the capillary number $Ca = \frac{\eta v}{\gamma}$ ($\eta$ is the fluid viscosity, $v$ is the average front velocity and $\gamma$ is the surface tension) is small. The displacement process

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of the fluid follows minimum resistance paths: the smaller pores are filled or invaded first.

Recently, many modifications of the original invasion percolation model were proposed. They take into account, for example, the action of an external gravitational field [3,4,5] or the flux with a privileged direction [6]. Many real-world applications of the invasion percolation model were found, extending from the economically important tertiary recovery of petroleum to the fingering phenomena in soils [7].

In the present paper, we study, in some detail, the effects arising from a process where invasion takes place simultaneously at several lattice sites. In the original formulation of the invasion percolation model [1], at each step of the growth, only one lattice site was allowed to be occupied. Here we permit a certain number of lattice sites to be invaded at the same time. This turns the model to be more realistic and close to the experiments. Of course, for such models, there are many possible generalizations. Henceforth, we shall call them generically as the multiple invasion percolation models. On the other hand, the original invasion percolation will be called the ordinary invasion percolation model. We restrict ourselves to the two-dimensional square lattice and to the site version of the invasion problem.

We propose three forms of multiple invasion percolation, which we have named models A, B and C. They were conceived, guided by some physical ingredients: the flux through the interface and the scaling relations between the mass and the gyration radius of fractal clusters. In the first model, the invasion is controlled by the size of the interface, i.e., by the size of the total perimeter. A parameter $F$ is introduced which corresponds to the fraction of the perimeter sites to be actually invaded. The second model is very close to the first, but the number of invaded sites is now controlled by the perimeter rectangle, i.e., by the rectangle inside which the invasion cluster is inscribed. Depending on the values of the parameter $F$, this procedure guarantees a continuous change of the fractal dimension $DF$ [8,9] in the interval $[\frac{91}{48},2]$. Finally, the third model is a more ambitious scheme where, at each step of the growth, the number of sites to be invaded is chosen in order that the resultant mass $M$ and gyration radius $R_g$ obey as best as possible the scaling law $M \sim R_g^{DF}$. The fractal dimension $DF$ here is a mere input parameter and can be varied from 0 to $\infty$. When $DF > 2$, an interesting burst phenomenon appears: most of the time the cluster grows as in the ordinary invasion (i.e., in each time step only one site or so is invaded) but suddenly, when the burst occurs, almost all available sites are occupied. For $DF < \frac{91}{48}$ the model is frustrated in the sense that it tries to invade less than one site, but it is inhibited by the algorithm. In the interval $[\frac{91}{48},2]$ the behavior follows the scaling law perfectly. We believe that the clusters generated in the latter form would be very useful to treat dilute magnetic systems [10] or random walks on random fractals [11]. The main reason is that for systems defined on usual random fractals the physical quantities are calculated only after the thermal and the cluster averages have been taken. For this cluster ensemble, the fractal dimensions vary wildly (between clusters or even for the same cluster
but at different sizes). These fluctuations render the physical parameters obtained in this way difficult or meaningless. Here such a situation is not reproduced because, by construction, our random clusters retain their fractal dimensions exactly the same all the time.

2. The perimeter model

There are actually two kinds of invasion percolation models: with and without trapping [1]. The trapping occurs when the displaced fluid is an incompressible fluid and it is completely surrounded by the other. These models fall into two different classes of universality. For the model with trapping $DF \sim 1.82$. The case without trapping corresponds to the critical ordinary percolation [1] ($DF = \frac{91}{48}$). In this paper, we treat only the model without trapping.

As pointed out by Grassberger and Manna [12] invasion percolation is a kind of self-organizing criticality [13] exhibiting scale-invariant behavior in time and space and evolving into a natural critical state. Before going to generalize the invasion percolation process to the multiple invasion, let us briefly recall the ordinary site invasion percolation algorithm:

- Assign random numbers $r$, uniformly distributed in the range $[0, 1]$, to each site of an $L \times L$ lattice;
- put a seed at the center of the lattice and define a list of growing sites composed by its first neighbors;
- the invading fluid occupies one growth perimeter site (that one with the lowest random number $r$);
- update the list of growing sites;
- the invasion process ends when the invading fluid reaches one boundary of the lattice.

The cluster generated in this way is connected to the critical ordinary site percolation [1]. Two simple ways of characterizing the cluster structures are to evaluate their fractal dimension and to evaluate their average coordination number. From the fact that the ordinary percolation can be seen in the limit $q \rightarrow 1$ of the $q$-states Potts model and that the latter can be solved exactly at the critical point (through the conformal invariance theory [14]), the fractal dimension $DF$ of the critical percolating cluster is known to be exactly $\frac{91}{48}$. On the other hand, as far as we know, the average coordination number $Z$ has been evaluated [15] only for the bond percolating cluster. So we proceed with the analysis of this quantity for the site problem.

The invaded sites of the invasion percolation cluster can be classified according to their number of first nearest neighbors occupied sites. Let $N_k$ be the number of sites surrounded by $k (k = 1, \ldots, 4$ for a square lattice) occupied sites. We calculate the fractions $N_k/\sum_k N_k$ and average them over 100–2000 simulations (the actual number of simulations is chosen in order to bring the statistical deviations down to 1%) on lattices of size $L = 101, 201, 401, 1201, 1601$ and 1801. The result is extrapolated using the BST algorithm [16] and it is presented in Table 1.
Table 1
The fractions $N_k/\sum_{k=1}^{4} N_k$ for the ordinary site invasion percolation. The estimated coordination number is $Z = 2.51 \pm 0.01$

<table>
<thead>
<tr>
<th>$N_1$</th>
<th>$N_2$</th>
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<tr>
<td>0.13</td>
<td>0.36</td>
<td>0.38</td>
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We see that sites of kinds $N_2$ and $N_3$ are dominant. Our estimated average coordination number is $Z = 2.51 \pm 0.01$, so the critical site percolating clusters are fractal lattices somewhere between the regular one-dimensional lattice ($Z = 2$) and the honeycomb lattice ($Z = 3$).

We consider now the first multiple site invasion percolation model which we will call model A. It seems reasonable that the invasion mechanism is, in some way, controlled by the flux through the surface. The set of empty sites (sites which were not invaded) of the square lattice outline one "sea" and many internal "lakes". Those sites situated at the frontier form, respectively, the external and internal perimeters. Then, one simple idea is to imagine that the invasion flux is proportional to the total perimeter. Let $F$ be $(0 \leq F \leq 1)$ the fraction of the perimeter sites to be invaded at each stage, i.e., if $N_p$ is the total number of perimeter sites (in some step of the growth), then $FN_p$ is the number of sites to be invaded (at the next step). With this definition, the number of invaded sites grows up together with the cluster and $F$ is a tuning parameter.

For $F \in [0, 1]$ and for several lattice sizes, we simulate this model evaluating the following mean values: the gyration radius $R_g$, the coordination number $Z$ and the mass $M$. The conclusion is that unless $F = 0$ the resulting cluster is compact! To understand what is going on, we plot in Fig. 1 the quantities $M/R_g^2$ and $Z$ versus $L$ for $F = 0.06$. We see that as we increase the lattice size, the coordination number $Z$ approaches 4 and $M/R_g^2$ goes to a constant. This means that the fractal dimension is equal to 2. We have generated many figures of these clusters and observed the following sequence: at the beginning the cluster has a form very similar to that in the ordinary invasion; as it grows, the internal lakes start disappearing; finally, only the external sea exists, the cluster is compact and it acquires a lozenge form. The conclusion is that, in the model A, the invasion rhythm is too fast and, for large enough lattices, the cluster is compact.

In order to overcome the difficulties exhibited by the model A, we need to decrease the number of perimeter sites involved in the growth process. One interesting way to do this is through the rectangle inside which the cluster is inscribed. The square root of its area can be used as a measure of the correlation length [17]. So, in the model B, we consider the number of sites to be invaded as being equal to $FN_r$, where $N_r$ is the number of sites contained in the perimeter of the rectangle.

For this model B, we classify the vertices types using the percentages $N_k/\sum_k N_k$ defined before. From Fig. 2(a) we see that $N_1$ and $N_2$ are monotonically decreasing
functions of $F$ whereas $N_3$ first increases slowly and then, near $F = 0.45$, has an abrupt fall. On the other hand, $N_4$ is always a monotonically increasing function of $F$ until $F \sim 0.5$. After this point, almost all vertices are of $N_4$ type. We also show in Fig. 2(b) how the average coordination number $Z$ depends on the parameter $F$. For $F > 0.5$ the model B generates compact clusters.

To analyze the growth mechanisms it is useful to define [1] the acceptance profile $a(r)$ which is the ratio of the number of random numbers in the interval $[r, r + dr]$ of sites accepted into the cluster to the number of random numbers available in that range. In the limit of an infinite lattice, the acceptance profile of the ordinary invasion percolation tends to a step function with the discontinuity located at the critical ordinary percolation threshold $p_c = 0.5928$ [18]. This fact, together with the estimated fractal dimension, is the essential argument used to connect the ordinary invasion percolation model with the critical ordinary percolation. So we performed the acceptance profile calculation as a function of $F$. In Fig. 3(a) we show some acceptance profiles when $F = 0.30$ for many lattice sizes. Again it approaches the step function but at a different critical random number: $r_c = 0.65$. The dependence of $r_c$ on $F$ is presented in Fig. 3(b). We conclude that the model B is an interpolation between the ordinary invasion and the completely compacted case in which all lattice sites are invaded.
Another interesting phenomenon occurring in the model B is related to the unusual gyration radius behavior. Normally, for a fixed lattice size $L$, one would expect that it augments with $F$ until $F = 0.5$ where it would attain the value $R_g = L/\sqrt{12}$, corresponding to a homogeneous square of side $L/\sqrt{2}$ (the side of the square inscribed...
Fig. 3. (a) Some acceptance profiles of the model B for fixed $F$ and various lattice sizes. (b) The critical threshold as a function of $F$.

in the lattice of size $L$). From Fig. 4(a) we see in the gyration radius a maximum around $F \sim 0.35$ ($L = 801$). We believe that this maximum may be connected to the fact that, for a given perimeter, the figure with the largest area is a circle. Indeed this is the form of the clusters (Fig. 4(b)). The forms of the clusters generated change
in the following way: for small values of $F$, the clusters are very similar to those of ordinary invasion, i.e., they have many branches and are clearly not homogeneous; for intermediate values of $F$, they become more uniform and acquire a circular form;
and, finally, for values greater than 0.5, they have the form of a compact square inscribed in the lattice.

3. The optimized model

Imagine that you are interested in studying some random model defined on a random cluster and suppose also that you are going to use computer simulations. All the physical quantities must be averaged over that cluster. Is that cluster representative? Usually the answer is no and then a second average over many clusters is necessary. From the point of view of the fractal dimension evaluation, this ensemble of clusters has usually very large fluctuations and worst, even for just one cluster, the fractal dimension depends strongly on when you stop growing it or, equivalently, on its size. The expected asymptotic dependences relating the mass and the gyration radius are, many times, in some forbidden computer region. These are the Gordian knots of any calculation of this kind. Thinking of these problems, we propose what we call the optimized model. The idea is to design a growth mechanism obeying exactly the scaling

\[ M \sim (R_g)^{DF} \]

or as near it as possible.

Let us now describe how the algorithm is built. At any invasion step \( j \), we always have at our disposal a list with the peripheral sites, i.e., a list with all the frontier sites ordered (in an increasing way) by their associated random numbers. The mass \( M(j-1) \) and the gyration radius \( R_g(j-1) \) at the anterior step \( j-1 \) are known. Suppose that this list has a size \( M_p \), the total number (mass) of perimeter sites. For each member of this list, say one at position \( k(1 \leq k \leq M_p) \), we can calculate its gyration radius \( R_g(j) \) if the \( k \) sites were invaded. The expected mass at step \( j, M_e(j) \), i.e., the mass coming from the scaling relation

\[ M_e(j) = M(j-1) \left[ \frac{R_g(j)}{R_g(j-1)} \right]^{DF} \]  

is normally different from the real mass \( M(j) = M(j-1) + k \). But we can search through the whole list for a \( k \) value for which

\[ ||M(j) - M_e(j)|| \]

is minimum. Observe that a nonstop growing process is guaranteed as long as \( k \geq 1 \). Here the fractal dimension \( DF \) plays only a role of an input parameter and can be chosen in the interval \((0, \infty)\).

The model described above has many extraordinary characteristics and it can exhibit new effects, particularly due to the possibility of extending the \( DF \) parameter. This will compete and sometimes conflict with the two-dimensional space where the lattice is embedded.
Fig. 5. The number of invaded sites against the time step of the optimized model for several values of $DF$: (a) 0.25; (b) 1.00; (c) 1.70 and (d) 1.89.

We recognize three different regimes: $0 < DF < \frac{91}{48}$, $\frac{91}{48} \leq DF \leq 2$ and $DF > 2$. In Fig. 5 we show the number of invaded sites ($NIS$) against time step ($t$) for $DF = 0.25, 1.00, 1.70, 1.89$ and a lattice of size $L = 201$. The time averages $\langle NIS \rangle$ are also indicated. When $DF = 0.25$ the cluster grows as in the ordinary invasion, i.e., at each time step only one site is invaded. The averages $\langle NIS \rangle$ increase as functions of $DF$ and the total time to touch the boundary decreases.

For $DF = 1.89$ we observe that although both the ordinary invasion and our model have the same fractal dimension, the invasion process is changed because $\langle NIS \rangle = 1$ (always) and $\langle NIS \rangle \sim 38$ (for $L = 201$, see further comment below). What is more impressive is how a given cluster evolves in different cases. In Fig. 6 the zigzag of the ordinary invasion percolation is compared with the beautiful straight line of our model. By construction, the fractal dimension is kept precisely constant during the entire period of growth of the cluster.

Let us now return to the point of different $\langle NIS \rangle$. Of course, to invade 1 or 38 sites does not make any sense when we take the thermodynamic limit. The important parameter here is the ratio of the number of actually invaded sites to the number of available sites at some step. These ratios are shown in Fig. 7, for $L = 201$ and $DF = 1.5, 1.92, 2.00, 2.5$, and indicate that they go to zero asymptotically only for $DF \leq 2!$ For $DF > 2$ something different is happening.
Fig. 6. The log–log plot of the mass as a function of the gyration radius for just one cluster grown on a lattice of size $L = 201$: (a) in the ordinary percolation; (b) in the optimized model.
We also studied the difference average $(M_e - M)(t) = \frac{\sum_{\tau=1}^{t} M_e(\tau) - M(\tau))}{t}$ between the expected mass $M_e$ and the real mass $M$ for several $DF$s. Typical results are shown in Fig. 8. For $DF = 5$, $M_e$ exceeds $M$ by a value around 3; for $DF = 0.25$, it is smaller by $\sim -1$; and for $DF = 1.92$, they are equal. The conclusion is the following: in the region $\frac{91}{48} \leq DF \leq 2$ our prescription works perfectly well and the resulting clusters possess the desired fractal dimension, but in the two regions $0 < DF < \frac{91}{48}$ and $DF > 2$ the system is frustrated! In the first region, the optimized model tends to invade less than one site but it is precluded by the algorithm itself, whereas, in the second region, it tends to invade more sites than a two-dimensional lattice permits. It is important to note that, in these two regions, our parameter $DF$ does not coincide with the fractal dimension. Only for $\frac{91}{48} \leq DF \leq 2$ is $DF$ really identified with the fractal dimension.

We would now like to have a better understanding of what happens in the region $DF > 2$. In Fig. 9(a) we plot the bulk mass $M$ versus $t$ for $DF = 5.00$ and $L = 201$. We see that time to time a discontinuity appears, exhibiting points with a large mass increment. In Fig. 9(b) we plot the gyration radius $R_g$ against the perimeter mass $M_p$. It shows some points where the expected augment of $M_p$ is reversed. These points correspond to the emerging of the burst phenomenon.
For $DF > 2$ the cluster development has the following scenario: during the majority of the steps the invasion is as in the ordinary case – only one site is invaded each time, but sudden explosions occur in which practically all available perimeter sites are occupied. Let us analyze, at some burst stage $j$, the behavior of the expected mass $M_e(j)$ and the really possible mass $M(j) = M(j - 1) + k$. To illustrate this point, we simulate $DF = 2.5$ in a lattice of size $L = 201$. We choose one burst that it is occurring at $t = 546$ (see Fig. 10) and which has $k = 478$ perimeter sites.

We see that, at the burst, there are two points where the curves $M_e$ and $M$ are very close to each other: one at the beginning of the list (list of the sites to be invaded) and the other at the end. Indeed, at the latter point they are closer to each other and it corresponds to the minimum of Eq. (3). So, at this stage, all available perimeter sites are invaded. In the next snapshot at $t = 547$, the situation has dramatically been changed – only in the beginning of the list $M_e$ and $M$ are very close and we indicate, in the figure, that only four sites are invaded this time.

Finally, we choose $DF = 5.00$ to illustrate the burst visually. It occurs at $t = 965$ and it is extraordinary. In Fig. 11 we show the cluster just before ($t = 964$) the burst. It is heterogeneous – one dense core surrounded by a sparse region. After the explosion, at $t = 966$, only the dense region exists.
Fig. 9. (a) The bursts seen as sudden mass jumps. (b) The gyration radius as a function of the perimeter mass $M_p$. When a burst takes place, many internal lakes disappear and the number of sites belonging to the perimeter decreases.
Fig. 10. (a) At the moment of the burst, the difference between the expected mass $M_e$ and the real mass $M$ is minimum if all 478 available sites are invaded; (b) the same plot but one time step later.
Fig. 11. For $DF = 5.00$ and $L = 201$ the burst is visually illustrated: (a) one time step before the burst; (b) one time step after the burst.
4. Conclusions

We studied three generalizations of the ordinary site invasion percolation. The first two, which we call models A and B, are based on the perimeter concept, i.e., they are controlled by the perimeter size. The model A, although with a strong physical appeal, revealed to be useless since it only generates compact clusters. The second, the model B, presents a continuous change in the fractal dimension. Varying the parameter $F$, the clusters evolve from a branched form very similar to the ordinary invasion, acquire an interesting circular shape and, finally, end up with a compact structure when $F \approx 0.5$. The acceptance profile analysis indicates that this model may correspond to some critical percolation model with the percolation threshold $p_c$ depending on $F$. The last model, which we call the optimized model, is based on scaling concepts. The parameter $DF$ can assume values from 0 to $\infty$. In the interval $(0, \frac{91}{45})$ the system is frustrated from below, i.e., the scaling relation points to an invasion of less than one site. On the other hand, in the region $DF > 2$, the system is frustrated from above and a very beautiful burst phenomenon takes place. In both cases, $DF$ does not coincide with the real fractal dimension. In the interval $[\frac{91}{45}, 2]$, the fractal dimension and $DF$ are the same and they do not oscillate when the cluster grows, i.e., the fractal dimension is a true and stable parameter not only in the asymptotic limit but also during all the time and scales. This last fact makes the model well suited to treat dilute systems where the fluctuations of the cluster ensembles overcast the reality.

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