Gaussian and Exponential GARCH models

Roberto N. Onody, G. M. Favaro, and Erike R. Cazaroto

1Instituto de Física de São Carlos, Universidade de São Paulo, CP 369, 13560-970, São Carlos, SP, Brazil

The GARCH(p,q) model is a very interesting stochastic process with widespread applications and a central role in empirical finance. The Markovian GARCH(1,1) model has only 3 control parameters and a much discussed question is how to estimate them when a series of some financial asset is given. The variance, the kurtosis and the autocorrelation time are the quantities actually used to determine them. Instead of the autocorrelation time, we propose here to use the standardized 6-th moment. A rapidly convergent series expansion can be established for it. The set of parameters obtained in this way produces a better probability density function and a much better time autocorrelation function. This is true regardless if the conditional probability density function in the GARCH process is Gaussian or Exponential. The probability of return to the origin is investigated at different time horizons for both Gaussian and Exponential GARCH models. In spite of the fact that these models show almost identical performances with respect to the final probability density function and to the time autocorrelation function, their scaling properties are, however, very different. The scaling exponent obtained by the Exponential GARCH model is very close to that of the real world.

I. INTRODUCTION

In recent years, physicists have shown an increasing interest in Economics problems [1]. The reason seems to be in the fact that many of these problems may be scrutinized by using standard tools of Statistical Physics [2].

In this paper, we investigate the generalized autoregressive conditional heteroscedasticity (GARCH) model [3, 4]. It was designed to describe a central question in theoretical and empirical finance - the volatility. Nowadays, the GARCH model plays an important role in the literature and it may be useful even in the complicated field of accurate forecasts [5].

The GARCH(p,q) model has (p+q+1) parameters. Here, we consider only the Markovian process GARCH(1,1). So, there are three parameters and they can be estimated by evaluating certain quantities of a financial asset. Usually, a Gaussian conditional probability density function is chosen for the GARCH process but many other distributions are possible. Gaussian GARCH models have the advantage that their variance, kurtosis and autocorrelation time are exactly known functions of the parameters. Clearly, by evaluating the variance, the kurtosis and the autocorrelation time of a financial asset one gets a good hint for the control parameters set. However, in the real world, to get a confident value for the autocorrelation time is very unlikely. Thus, one of the parameters is arbitrarily chosen in order to give a very large autocorrelation time. Although this procedure gives reasonable results for the final probability density function (PDF), it fails to reproduce correctly the time dependence of the autocorrelation function. The reason is quite simple: financial assets have autocorrelation function decaying as power law while the Gaussian GARCH decays exponentially. Worth to say here, that we are speaking of the autocorrelation time obtained from the autocorrelation function of the square of the return.

Instead of the autocorrelation time, we propose the use of the standardized 6-th moment (i.e., the 6-th moment divided by the cubic of the variance). Regrettably, even for the Gaussian case, no exact expression is available for it. But one can write down a series expansion for the 6-th moment. We shall show that this series rapidly converges and a set of extrapolated parameters of the GARCH model can be calculated. Simulations of the GARCH model, with this set of parameters, reveal a much better agreement with the autocorrelation function of a real asset. Moreover, this characteristic is robust, i.e., it is preserved even when the Gaussian is substituted by an exponential distribution.

We also study the Exponential GARCH model, that is, the GARCH model with the conditional PDF decaying exponentially with the return. We derive an exact expression for the kurtosis, which is written in terms of the GARCH parameters. But, just like in the Gaussian case, a series expansion is necessary for the standardized 6-th moment.

The performances of the Gaussian and the Exponential GARCH models are then compared when they both simulate the time evolution of the NYSE (New York Stock Exchange) composite index. With respect to the PDF and to the time autocorrelation function, both models are practically equivalents, giving fairly good results. Their differences only appear when the scaling properties are studied. To test the effectiveness of the models, we need to investigate how the probability of return to the origin scales for any time horizon. It is well known that this is the Achilles heel of the Gaussian GARCH model [1]. However, the Exponential GARCH model works...
fine. For the NYSE index, we estimated a scaling exponent $0.691 \pm 0.062$ whereas, for the Exponential GARCH, we found $0.635 \pm 0.022$. So they agree within the error bars. In the next section, we present a brief review of the ARCH and GARCH models.

II. ARCH AND GARCH MODELS

An ARCH model is a stochastic process with autoregressive conditional heteroscedasticity. It was first introduced by Engle in 1982 [6]. ARCH models are simple models capable to describe a stochastic process which is locally non-stationary but asymptotically stationary. If the stochastic process exhibits a time dependent variance, i.e., volatility, then the ARCH models are particularly useful and wherefore have been applied to many different areas of economics: interest rates, stock returns, foreign exchange rates, etc. In an ARCH process, the variance at a time $t$ depends on some past values and it is characterized by a certain number of parameters. An ARCH(p) process is defined by the equation

$$
\sigma_t^2 = \alpha_0 + \alpha_1 x_{t-1}^2 + \ldots + \alpha_p x_{t-p}^2,
$$

where the parameters $\alpha_0, \alpha_1, \ldots, \alpha_p$ are positive constants and $x_t$ is a random variable, with zero mean and variance $\sigma_t^2$, coming from some conditional probability density function $P_t(x_t)$. Once the parameters $\alpha$'s and the form of $P_t(x_t)$ of an ARCH(p) model are chosen, eq. (1) is iterated and the asymptotic distribution of $x_t$ is determined and compared with the probability density function of some financial asset. Unfortunately, in order to get good results, ARCH(p) models need very long memories (large $p$). For this reason, Bollerslev [3] proposed in 1986 a generalization of the ARCH’s models, the GARCH(p,q) processes. A GARCH(p,q) model is defined by

$$
\sigma_t^2 = \alpha_0 + \alpha_1 x_{t-1}^2 + \ldots + \alpha_p x_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \ldots + \beta_q \sigma_{t-q}^2
$$

Here, the $\alpha$'s and $\beta$'s are control parameters (all real positive constants) and $x_t$ are random variables with zero mean and variance $\sigma_t^2$ obtained from a conditional probability distribution $P_t(x_t)$, usually taken to be Gaussian. In this paper, we restrict our analysis to the Markovian process GARCH(1,1)

$$
\sigma_t^2 = \alpha_0 + \alpha_1 x_{t-1}^2 + \beta_1 \sigma_{t-1}^2,
$$

which has only 3 parameters $\alpha_0, \alpha_1$ and $\beta_1$. The initial condition is assumed to be $\sigma_0^2 = 0$.

The analytical expression of the n-th moment after $T$ iterations is given by

$$
<x^n> = \int \ldots \int \prod_{t=1}^{T-1} P_t(x_t)dx_t \int P_T(x_T)x_T^n dx_T
$$

Because the distribution $P_t(x_t)$ has zero mean the variance is equal to the second moment. Besides the 2-th moment, we can calculate the 4-th and 6-th moments. These moments, in a GARCH(1,1) process, are functions of the parameters $\alpha_0, \alpha_1$ and $\beta_1$, so if we made them equal to the corresponding moments of some financial asset, we remain with 3 equations for those 3 parameters. The solution of these 3 equations gives us the best parameters values, i.e., the parameters which bring the GARCH dynamics as close as possible to the real economic process. However, this approach of using the 2-th, 4-th and 6-th moments, does not correspond to the standard procedure used in the literature (see below).

Last but not least, it is important to determine the autocorrelation functions in the GARCH dynamics. The autocorrelation $<x_t x_{t+\tau}>$ of the random variable $x_t$ is proportional to a delta function $\delta(\tau)$. Consequently, only higher-order correlations are interesting or useful. In particular, the autocorrelation for the $x_t^2$ variable, we found $0.062$ whereas, for the Exponential GARCH, $0.22$. So they agree within the error bars. For the NYSE index, we estimated a scaling exponent $0.691 \pm 0.062$ whereas, for the Exponential GARCH, we found $0.635 \pm 0.022$. So they agree within the error bars. In the next section, we present a brief review of the ARCH and GARCH models.

Because the distribution $P_t(x_t)$ has zero mean the variance is equal to the second moment. Besides the 2-th moment, we can calculate the 4-th and 6-th moments. These moments, in a GARCH(1,1) process, are functions of the parameters $\alpha_0, \alpha_1$ and $\beta_1$, so if we made them equal to the corresponding moments of some financial asset, we remain with 3 equations for those 3 parameters. The solution of these 3 equations gives us the best parameters values, i.e., the parameters which bring the GARCH dynamics as close as possible to the real economic process. However, this approach of using the 2-th, 4-th and 6-th moments, does not correspond to the standard procedure used in the literature (see below).

Last but not least, it is important to determine the autocorrelation functions in the GARCH dynamics. The autocorrelation $<x_t x_{t+\tau}>$ of the random variable $x_t$ is proportional to a delta function $\delta(\tau)$. Consequently, only higher-order correlations are interesting or useful. In particular, the autocorrelation for the $x_t^2$ variable, we found $0.062$ whereas, for the Exponential GARCH, $0.22$. So they agree within the error bars. For the NYSE index, we estimated a scaling exponent $0.691 \pm 0.062$ whereas, for the Exponential GARCH, we found $0.635 \pm 0.022$. So they agree within the error bars. In the next section, we present a brief review of the ARCH and GARCH models.

$$
< x_t^2 > = \int \ldots \int \prod_{t=1}^{T-1} P_t(x_t)dx_t \int P_T(x_T)x_T^2 dx_T
$$

$$
F(\tau) = \frac{< x_t^2 x_{t+\tau} > - < x_t^2 > < x_{t+\tau}^2 >}{< x_t^2 > - (< x_t^2 >)^2}
$$

III. GAUSSIAN GARCH MODELS

A model is said a Gaussian GARCH model if the conditional probability function $P_t(x_t)$ is Gaussian with fluctuating variance. More explicitly

$$
P_t(x_t) = \frac{\exp(-\frac{x_t^2}{2\sigma_t^2})}{\sigma_t(2\pi)^\frac{1}{2}},
$$
As we shall see, the great obstacle is the autocorrelation 

\[ \sigma^2(\alpha_0, \alpha_1, \beta_1) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}, \tag{10} \]

inasmuch as \((\alpha_1 + \beta_1) < 1\). Indeed, with this restriction, eq.(10) holds for any probability density function.

The kurtosis \(\kappa\) is defined as the 4-th moment divided by the square of the variance. For a Gaussian distribution, the kurtosis can be immediately determined:

\[ \kappa_G(\alpha_1, \beta_1) = \frac{<x^4>}{\sigma^4} = 3 + \frac{6\alpha_1^2}{1 - 3\alpha_1^2 - 2\alpha_1\beta_1 - \beta_1^2} \tag{11} \]

This means that if the autocorrelation time, the variance and the kurtosis are determined for some financial asset, then the equations (7), (10) and (11) would provide solutions for the 3 GARCH’s parameters \(\alpha_0, \alpha_1\) and \(\beta_1\). As we shall see, the great obstacle is the autocorrelation time \(\tau_c\). Its determination is very difficult and imprecise. The point is that, in the real world, ensemble averages are not possible so a good estimation of \(\tau_c\) is out of question.

To see this, we analyze the daily values of the NYSE (New York Stock Exchange), recorded from December 31, 1965 to January 31, 2006 (10,088 points). Let \(Y(t)\) be the NYSE composite index at time \(t\). As our frequency data is very low, the recommended random variable to be used here is the return \(Z_{\Delta t}(t)\) defined by

\[ Z_{\Delta t}(t) = \ln(Y(t + \Delta t)) - \ln(Y(t)) \tag{12} \]

Hereafter, for the sake of simplicity, we will denote \(Z(t) \equiv Z_{\Delta t=1}(t)\) when \(\Delta t = 1\).

For the NYSE composite index, the variance and kurtosis can be immediately determined: \(s_{\text{nyse}} = 8.084 \times 10^{-6}\) and \(\kappa_{\text{nyse}} = 38.507\). On the other hand, the time autocorrelation function \(F(\tau)\) has the very wild behavior shown in the Figure 3. An estimative of \(\tau_c\) can be obtained by finding that time in which \(F(\tau)\) turns negative for the first time. We get \(\tau_{c,\text{nyse}} = 143\).

Substituting the values above into equations (7), (10) and (11), we calculate the GARCH set of parameters \(s_1\)

\[ s_1 = (\alpha_0, \alpha_1, \beta_1) = (5.63 \times 10^{-7}, 5.12 \times 10^{-2}, 0.94185) \tag{13} \]

Values of \(\beta_1\), bigger than 0.9, are often used in the literature, e.g., \(\beta_1 = 0.90000\) for the S&P 500 [1] and \(\beta_1 = 0.90501\) for the stock prices of the Center for Research in Security Prices (CRSP) [8].

We simulate GARCH processes using this control parameters set \(s_1\). As for the NYSE series, the basic random variable here is the return defined in the eq.(12) but with the NYSE index \(Y(t)\) replaced by the variable \(x_t\). The series of \(x_t\) is built by iterating the stochastic equation (3).

Although our simulations give an asymptotic PDF (probability density function) which is in good agreement with that of the NYSE data (see Fig.2), the same cannot be said on the autocorrelation function. As can be seen from the Fig.3, the GARCH autocorrelation function \(F(\tau)\), averaged in an ensemble of 10,000 runs, is very far from the corresponding NYSE function.

To overcome this difficulty, instead of using the time autocorrelation \(\tau_c\), we propose the standardized 6-th moment \(\Theta = \frac{x^6}{\sigma^6}\). Using eq.(10), this quantity can be written as a function of only 2 parameters, i.e., \(\Theta = \Theta(\alpha_1, \beta_1)\).

Unfortunately, an analytical expression for \(\Theta(\alpha_1, \beta_1)\) does not exist. But, from eq.(4), one can write down a two-variables series expansion for \(\Theta(\alpha_1, \beta_1)\). The fundamental point is that this series is exact in \(\alpha^{m} \beta^{n}\) up to order \((m + n) = 2T - 1\), where \(T\) is the number of iterations of eq.(4). Here, by exact, we mean that further iterations will not alter the actual coefficients up to order \(2T - 1\). So we can write \(\Theta(\alpha_1, \beta_1, T)\) such that \(\Theta(\alpha_1, \beta_1, \infty) = \lim_{T \to \infty} \Theta(\alpha_1, \beta_1, T)\).

Using Maple [9], we were able to go up to \(T = 30\). Thus there is a sequence of \(\Theta(\alpha_1, \beta_1, T), T = 1, \ldots, 30\), which can then be explored by extrapolation techniques. For the NYSE series, we have \(\kappa_{\text{nyse}} = 38.507\) and \(\Theta_{\text{nyse}} = 17717.37\). Solving the kurtosis equation \(\kappa_G(\alpha_1, \beta_1) = 38.507\) and the sequence of \(\Theta_G(\alpha_1, \beta_1, T) = 17717.37\), we obtain curves \(\beta_1(\alpha_1)\) plotted in the Figure 1. Where these curves cross the kurtosis curve, gives a sequence of solutions \(\alpha_1(T)\) which are shown in the inset. The data are very well fitted by the exponential
where $B = 0.82 \pm 0.04$, $\Gamma = 9.47 \pm 0.52$ and $C = 0.26391 \pm 0.00574$. So, the extrapolated value of $\alpha_1$ is 0.26391.

Using equations (10) and (11), we arrive to a new set $s_2$ for the control parameters

$$s_2 = (\alpha_0, \alpha_1, \beta_1) = (7.81 \times 10^{-6}, 0.26391, 0.63947) \quad (15)$$

We can now simulate Gaussian GARCH processes using the two sets of parameters. In the Figure 2, we plot the PDF of the NYSE composite index together with 10,088 points generated by GARCH’s dynamics with both sets $s_1$ and $s_2$. The GARCH PDF’s are less leptokurtic than NYSE PDF. Let us denote their PDF’s by $H_{\text{nyse}}(Z)$, $H_{s_1}(Z)$ and $H_{s_2}(Z)$. We observe that if $|Z| > 0.005$ then $H_{s_1}(Z) > H_{s_2}(Z) > H_{\text{nyse}}(Z)$ and the other way around, if $|Z| < 0.005$ than $H_{s_1}(Z) < H_{s_2}(Z) < H_{\text{nyse}}(Z)$. So this means that the set $s_2$ gives better results than $s_1$, as it is always more close to $H_{\text{nyse}}(Z)$.

What can we say about the autocorrelation function $F(\tau)$? Which set of parameters gives the best results? It is clear from the Figure 3 that, again, the set $s_2$ is superior than $s_1$ and in this case, much superior.

We conclude that, in spite of the its wide use in the literature, $s_1$ is not the best set of GARCH parameters, our proposal - of using the standardized 6-th moment instead the time autocorrelation - produces a set of parameters $s_2$ which fits much better both the probability density function and the autocorrelation function of the NYSE index. Particularly, for the autocorrelation function, the differences between the two sets are very impressive. In the next section, we develop a similar analysis for the Exponential GARCH model and we arrive to the same conclusion - the standardized 6-th moment parameters performs better than the time autocorrelation.

**IV. EXPOENTIAL GARCH MODELS**

Instead of using a Gaussian conditional probability as in the previous section, we propose an exponential with the form

$$P_t(x_t) = \frac{\exp(-\sqrt{2|x_t|}}{\sigma_t \sqrt{2}}, \quad (16)$$

where the variance $\sigma_t^2$ changes with time according eq. (3).

As for the Gaussian case, we investigate the moments which can be derived from eq.(4). The second moment is given once more by the eq.(10), but the kurtosis relation is quite different. We deduced the following analytic expression for the kurtosis of an Exponential GARCH process

$$\kappa_E(\alpha_1, \beta_1) = \frac{x^4}{\sigma^4} = 6 + 30\alpha_1^2 - \frac{6(1 - 3\alpha_1^2 - 2\alpha_1\beta_1 - \beta_1^2)}{1 - 6\alpha_1^2 - 2\alpha_1\beta_1 - \beta_1^2} \quad (17)$$

Using the variance, the kurtosis and the time autocorrelation, that is, equations (10), (17) and (7), we determine the following set of parameters

$$ss_1 = (\alpha_0, \alpha_1, \beta_1) = (5.63 \times 10^{-7}, 0.04842, 0.94461) \quad (18)$$
For the standardized 6-th moment $\Theta(\alpha_1, \beta_1)$, one can write down a series expansion using simultaneously equations (4), (16) and (3). Once again, this series turns to be exact up to order $(2T - 1)$.

neighbors at the time autocorrelation function in the best possible

Using equations (10) and (17), we arrive to a new set $ss_2$ for the control parameters

$$ss_2 = (\alpha_0, \alpha_1, \beta_1) = (4.73 \times 10^{-6}, 0.13847, 0.80303) \quad (19)$$

As in the previous section, we can now simulate Exponential GARCH processes using the two sets of parameters. In the Figure 5, we plot the PDF of the NYSE composite index together with 10,088 points generated by GARCH’s dynamics with both sets $ss_1$ and $ss_2$. Contrary to what happens in the Gaussian case, here the GARCH PDF’s are more leptokurtic than the NYSE PDF. The $ss_2$ set of parameters seems to be a little bit closer to the NYSE curve than the $ss_1$ set.

This difference becomes much stronger when we look at the time autocorrelation function $F(\tau)$. This is clearly shown in the Figure 6.

So, no matter we are dealing with Gaussian or Exponential GARCH models, the use of the standardized 6-th moment unequivocally gives better results than the use of the time autocorrelation. As a final comment, we would like to point out how far are the parameters $\beta_1$ we got, $\beta_1 = 0.64$ and $\beta_1 = 0.80$ (for the Gaussian and Exponential GARCH, respectively) from those suggested in the literature [1], $\beta_1 > 0.9$.

Although the prescription that we have proposed here really improves the performance of a GARCH process, still remains an open question if this precise description of the time evolution of a financial asset scales correctly at any time horizon $\Delta t$. This is the subject of the next section.

V. SCALING PROPERTIES

First, let us compare the capabilities of a Gaussian and an Exponential GARCH process to describe the PDF’s and the time autocorrelation function in the best possible
scenario, i.e., using the set of parameters coming from the standardized 6-th moment. In the Figure 7, we show their PDF’s together with that of the NYSE index. The Gaussian (Exponential) is less (more) leptokurtic than the NYSE data. From the figure one can say that they are even.

![FIG. 7: The PDF's versus the returns Z. The Gaussian as well as the Exponential GARCH PDF's were obtained using the set of parameters s2 and ss2.](image)

There is again no winner if the time autocorrelation function $F(\tau)$ is analyzed (Figure 8). To settle the dispute, we need to investigate their scaling properties.

![FIG. 8: The time autocorrelation function $F(\tau)$ versus time $\tau$ for the NYSE, Gaussian and Exponential GARCH processes.](image)

By using equation (12), we can determine, for each one of them, the probability of return to the origin $P(0)$. This quantity scales as a power law

$$P(0) \sim (\Delta t)^{-\alpha} \quad (20)$$

By changing the time horizon $\Delta t$, one can answer whether the overall dynamics is well described by a GARCH(1,1) process. In the Figure 9, we estimate the exponents $\alpha$. As expected, the Gaussian GARCH gives the Gaussian exponent $\alpha = 0.499 \pm 0.001$ and it is far from the real NYSE value $\alpha = 0.691 \pm 0.062$. However, the Exponential GARCH model has $\alpha = 0.635 \pm 0.022$. This value is very near to the NYSE exponent and within the error bars.

![FIG. 9: The probability of return to the origin versus the time horizon $\Delta t$ for the NYSE, Gaussian and Exponential GARCH processes.](image)

**VI. CONCLUSIONS**

We have demonstrated that the use of the autocorrelation time to predict the best set of GARCH’s control parameters is not a good choice. Better results came out if the standardized 6-th moment is used instead. Despite the fact that the final PDF is only a little bit more close to the real data, if the 6-th moment is adopted, the gain in the time autocorrelation function is much more impressive. This result is robust and applies to the GARCH model irrespective if the conditional probability density function utilized is Gaussian or Exponential. Although, in both cases, a series expansion was necessary for the 6-th moment, we were able to derive an exact expression for the kurtosis of the Exponential GARCH model. A comparison between the Gaussian and Exponential forms show that they are practically equivalent with respect to the final PDF and the time autocorrelation function but they are not equivalent regarding their scaling properties.

When the probability of return to the origin is analyzed, the Exponential GARCH model performs much better than its Gaussian counterpart and gives a scaling exponent which is in a fairly good agreement with that of real data.

The authors are very grateful to CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) and CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior) for the financial support.


[9] Maple is a registered trademark of Waterloo Maple Inc.